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CLEAN TRIANGULATIONS

NORA HARTSFIELD and GERHARD RINGEL

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A polyhedron on a surface is called a clean triangulation if each face is a triangle and each triangle is a face. Let S_p (resp. N_p) be the closed orientable (resp. nonorientable) surface of genus p. If $\tau(S)$ is the smallest possible number of triangles in a clean triangulation of S, the results are: $\tau(N_1)=20$, $\tau(S_1)=24$, $\lim \tau(S_p)p^{-1}=4$, $\lim \tau(N_p)p^{-1}=2$ for $p\to\infty$.

1. Introduction

A polyhedron on a closed surface S is called a *triangulation* if each face is a triangle with three distinct vertices, and the intersection of any two distinct triangles is either empty, a single vertex, or a single edge. In other words, a triangulation of S is a 2-cell embedding of a graph into S where every 2-cell is a triangle. A triangulation is called *minimal* if the number of triangles is minimal. We denote the number of triangles in a minimal triangulation of S by $\delta(S)$.

Let S_p (resp. N_q) denote the orientable (resp. nonorientable) surface of genus p (resp. q). Jungerman and Ringel [4] have shown that

$$\delta(S_p)=2\left\lceil\frac{7+\sqrt{1+48p}}{2}\right\rceil+4(p-1)\quad\text{if}\quad p\neq 2 \tag{1}$$

and

$$\delta(S_2)=24.$$

Ringel [6] has shown that the same formula holds for N_q if we replace p by $\frac{q}{2}$ in (1), except when q is equal to 2 or 3, and

$$\delta(N_2)=16,\quad \delta(N_3)=20.$$

The 1-skeleton of a triangulation is a graph, and if this graph has a cycle of length 3 which is not the boundary of a 2-cell in the embedding, we call this cycle an *extra triangle*. If a triangulation has no extra triangles, we call it a *clean triangulation*. In other words, in a clean triangulation, not only is every face a triangle, but also every triangle is a face. A clean triangulation of a surface S is called *minimal* if the number of triangles is minimal. We denote the number of triangles in a minimal clean

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triangulation of S by $\tau(S)$. The tetrahedron, which is the minimal triangulation of the sphere, is also a clean triangulation, hence in this one instance $\delta(S_0) = \tau(S_0)$.

We find it very surprising that it seems to be impossible to derive a good lower bound for $\tau(S)$ using Euler's polyhedral formula.

We shall prove that

$$\tau(N_1) = 20, \quad \tau(S_1) = 24,$$

and we conjecture that $\tau(N_2) = 28$. If we compare these values with the δ values for the same surface, we find

$$\delta(N_1) = 10, \quad \delta(S_1) = 14, \quad \text{and} \quad \delta(N_2) = 16.$$

The values of τ seem to be significantly larger than the values of δ for the same surface. But we have another surprise. We shall prove that

$$\lim_{p\to\infty} \frac{\tau(S_p)}{p} = 4 \quad \text{and} \quad \lim_{q\to\infty} \frac{\tau(N_q)}{q} = 2.$$

Thus $\tau(S)$ exhibits the same asymptotic behavior as $\delta(S)$.

Tutte [7] considered clean triangulations also. He derived an asymptotic formula for the number of clean triangulations of the plane.

2. Low Order Cases

Theorem 1. $\tau(S_1) = 24$ *and Fig. 1 illustrates the only minimal clean triangulation of the torus.*

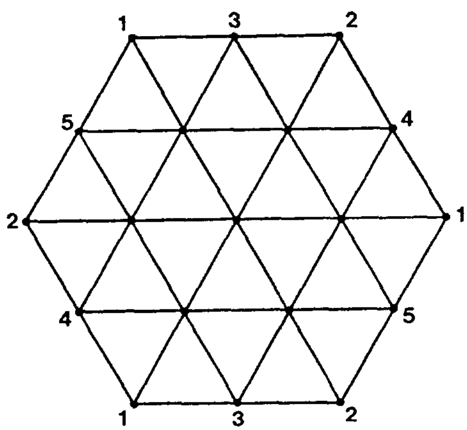

Fig. 1. Minimal clean triangulation of the torus

Proof. Suppose that we have a minimal clean triangulation of S_1 with V vertices, E edges, and F triangles. If we assume that the triangulation is regular of degree k, since F is equal to $2/3E$, by Euler's formula we find that k must equal 6.

Suppose that x is any vertex in the clean triangulation of S_1. Consider the triangles at distance at most 1 from x. (*See Fig. 1*, disregarding the numbers.)

There are 24 such triangles. If any two of the triangles are the same, since all vertices are at distance at most 2 from x, there will be an extra triangle. Thus if the triangulation is regular, there are at least 24 triangles.

Now assume that the triangulation is not regular. Suppose that the vertex of greatest degree has degree 7, and suppose further that $\tau(S_1) \leq 22$. Since $3F = 2E$, we know that F is even. Let x be a vertex of degree 7. Let 1, 2, ..., 7 be the vertices adjacent to x, and let a_i denote the number of triangles on which vertex i lies. Clearly, a_i is equal to the degree of vertex i. Then we have

$$a_1 + a_2 + \cdots + a_7 - 14$$

triangles at a distance at most 1 from x. We subtracted 14 because in the sum of the a_i's 14 triangles are counted twice, namely those triangles which have 2 vertices in common from 1, 2, ..., 7. It follows that

$$a_1 + a_2 + \cdots + a_7 - 14 \leq 22$$

or

$$a_1 + a_2 + \cdots + a_7 \leq 36. \tag{2}$$

If we assume that $F = 22$, we know $V = 11$ and $E = 33$. If we let b_1, b_2 and b_3 denote the degrees of the other three vertices we have

$$a_1 + a_2 + \cdots + a_7 + 7 + b_1 + b_2 + b_3 = 66. \tag{3}$$

Combining (2) and (3) we obtain

$$b_1 + b_2 + b_3 \geq 23,$$

which contradicts the assumption that 7 is the largest degree.

It is clear that assuming that we have a vertex of degree greater than 7 makes things even worse. Thus

$$\tau(S_1) \geq 24.$$

Now look at *Fig. 1* again and identify edges which have the same labels on both endpoints in the obvious way. The result is a clean triangulation of the torus with 24 triangles. Hence $\tau(S_1) \leq 24$ and therefore $\tau(S_1) = 24$. We omit the proof of the uniqueness of the minimal clean triangulation in *Fig. 1* since the proof is too elaborate. ∎

Theorem 2. $\tau(N_1) = 20$.

Proof. Suppose that there exists a clean triangulation of the projective plane with only 18 triangles. There is at least one vertex of degree greater than or equal to 6. Let x denote the vertex of largest degree, and assume the degree of x is 6. Let 1, 2, ..., 6 be the vertices adjacent to x and let a_i denote the degree of vertex i. If F is equal to 18, then Euler's formula tells us that E is equal to 27 and V is equal to 10. If we count the triangles at distance at most 1 from x, we have

$$a_1 + a_2 + \cdots + a_6 - 12 \leq 18.$$

or

$$a_1 + a_2 + \cdots + a_6 \leq 30. \tag{4}$$

If b_1, b_2, b_3 are the degrees of the other three vertices in the triangulation, then

$$a_1 + a_2 + \cdots + a_6 + 6 + b_1 + b_2 + b_3 = 54. \tag{5}$$

Combining (4) and (5) we obtain

$$b_1 + b_2 + b_3 \geq 18$$

Now we consider two cases.
Case 1.

$$a_1 + a_2 + \cdots + a_6 \leq 29.$$

In this case we have

$$b_1 + b_2 + b_3 \geq 19$$

which contradicts the choice of x as the vertex of largest degree.
Case 2.
If

$$a_1 + a_2 + \cdots + a_6 = 30,$$

then there are several possibilities for the a_i's with

$$4 \leq a_i \leq 6.$$

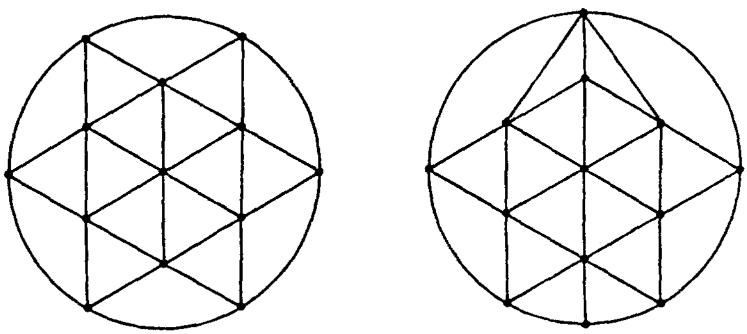

Fig. 2

Two of these possibilities are pictured in *Fig. 2*. In any of these we find there are six "outside vertices" which must be identified in such a way as to produce three vertices. The only way to identify without creating loops or double edges is to identify opposite vertices, but the outside edges then form an extra triangle.

Since assuming x has degree greater than 6 makes the argument even stronger, we have shown that

$$\tau(N_1) \geq 20.$$

Fig. 3 shows a clean triangulation of N_1 with 20 triangles and thus

$$\tau(N_1) = 20. \qquad \blacksquare$$

Fig. 4 shows a clean triangulation of Klein's bottle with 28 triangles. We strongly believe it is minimal.

In order to prove the main result of the paper, we need some preliminary results. An embedding of a graph is a *quadrangular embedding* if each face of the embedding is a quadrangle.

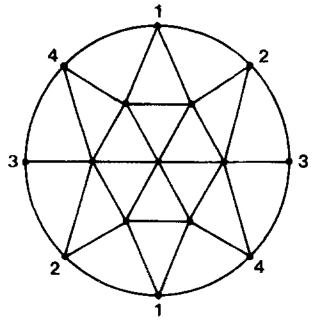

Fig. 3. Minimal clean triangulation of the projective plane

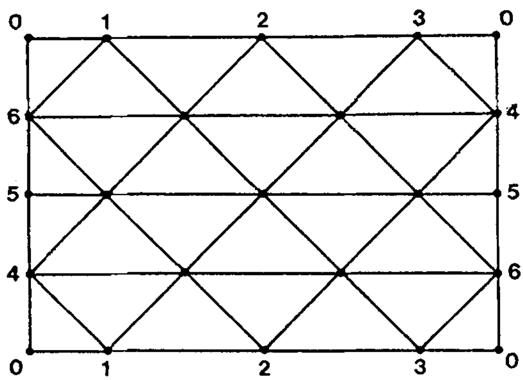

Fig. 4. Clean triangulation of Klein's bottle

3. Quadrangular Embeddings of $\mathbf{K}_{2n,2n}$.

Ringel [5] determined that the genus of the complete bipartite graph $K_{m,n}$ is

$$\left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

If both m and n are even, the enclosed quantity is an integer, and the genus embeddings of $K_{m,n}$ are quadrangular embeddings. Here we shall use his embeddings for the special case of $K_{2n,2n}$. To construct clean triangulations, we need some special properties of these embeddings.

The graph $K_{2n,2n}$ is presented in the following way. We denote the vertices by

$$0, 1, 2, 3, \dots, 4n-1,$$

and two vertices are adjacent if and only if the numbers have different parity.

Fig. 5 shows a quadrangular embedding of $K_{6,6}$ in S_4. One should identify two edges if their endpoints are both labeled with the same pair of numbers. Then we obtain a closed orientable surface, and we can easily determine using Euler's formula that it is S_4. In general the embedding of $K_{2n,2n}$ is into S_{n^2-2n+1}.

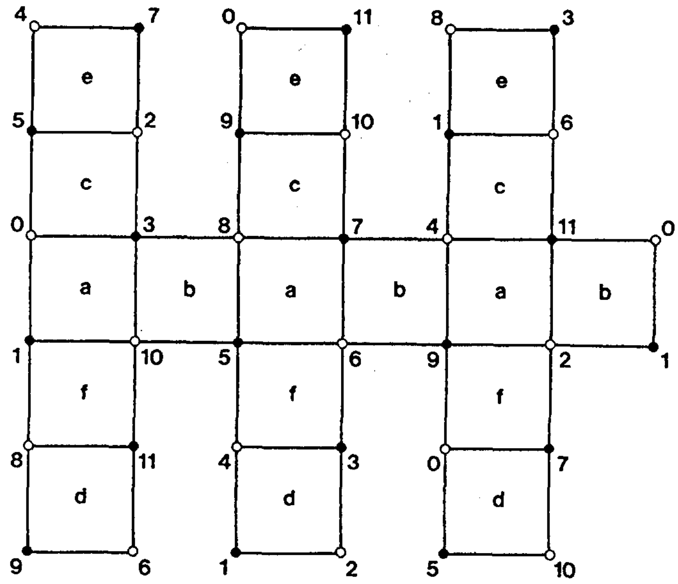

Fig. 5. Quadratic embedding of $K_{6,6}$ into S_4.

Referring to *Fig. 5,* we can read off the neighbors of vertex 0 as they appear in the surface counterclockwise around vertex 0,

$$0.1, 3, 5, 7, 9, 11.$$

We can do this for each vertex, thus obtaining a *combinatorial scheme* for the embedding. The scheme for the embedding in *Fig. 5* is

0.	1, 3, 5, 7, 9, 11	1.	0, 2, 4, 6, 8, 10
2.	11, 9, 7, 5, 3, 1	3.	10, 8, 6, 4, 2, 0
4.	1, 3, 5, 7, 9, 11	5.	0, 2, 4, 6, 8, 10
6.	11, 9, 7, 5, 3, 1	7.	10, 8, 6, 4, 2, 0
8.	1, 3, 5, 7, 9, 11	9.	0, 2, 4, 6, 8, 10
10.	11, 9, 7, 5, 3, 1	11.	10, 8, 6, 4, 2, 0

Since the embedding is a quadrangular embedding, this scheme satisfies the following rule.
Rule Q. If in rows i and k we have

$$\begin{array}{ll} i. & \ldots j, k, \ldots \\ k. & \ldots i, l, \ldots \end{array}$$

then in rows j and l we have

$$\begin{array}{ll} j. & \ldots l, i, \ldots \\ l. & \ldots k, j, \ldots . \end{array}$$

If C is a combinatorial scheme of a graph G which satisfies rule Q, then there exists a quadrangular embedding of G into an orientable surface, and the scheme of the embedding is precisely C (Edmonds' Technique) [3].

The generalization of the scheme for an embedding of $K_{2n,2n}$ into S_{n^2-2n+1} is obvious. The first four lines are

$$\begin{array}{llll} 0. & 1,3,5,\ldots,4n-1 & 1. & 0,2,4,\ldots,4n-2 \\ 2. & 4n-1,\ldots,5,3,1 & 3. & 4n-2,\ldots,4,2,0. \end{array}$$

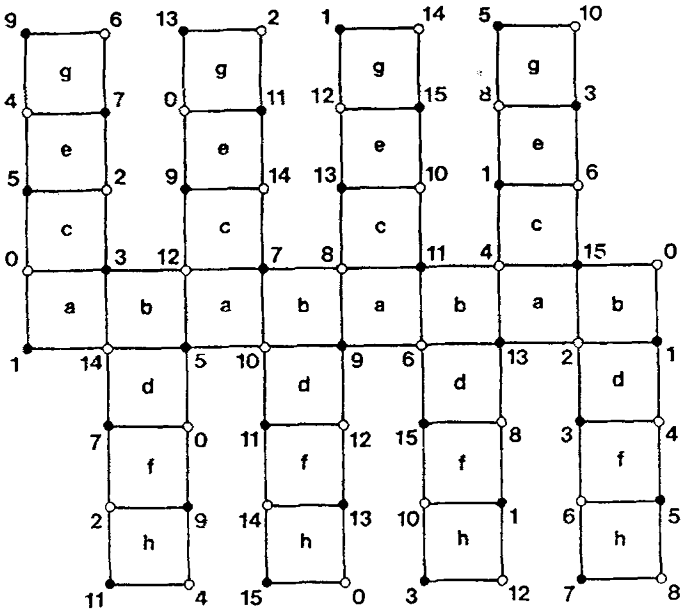

Fig. 6. Quadrangular embedding of $K_{8,8}$ into S_9.

Fig. 6 shows the case $n = 4$.

We shall call the faces of this embedding *squares.* There are $2n^2$ such squares. This embedding possesses the pleasant property that one can select $2n$ classes of n squares, each such that in each class all $4n$ vertices appear, each exactly once. In other words, each class represents every vertex. Thus we shall call them *representative classes.* In *Figs. 5 and 6* the classes are designated by the letters a, b, c, ...

In the case $n = 3$, the classes can be listed by the following double rows.

$$\begin{array}{ccccccccccccc} 0 & & 3 & & 8 & & 7 & & 4 & & 11 & & 0 \\ & a & & b & & a & & b & & a & & b & \\ 1 & & 10 & & 5 & & 6 & & 9 & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccccccccc} 4 & & 3 & & 0 & & 7 & & 8 & & 11 & & 4 \\ & d & & c & & d & & c & & d & & c & \\ 1 & & 2 & & 5 & & 10 & & 9 & & 6 & & 1 \end{array}$$

$$\begin{array}{ccccccccccccc} 8 & & 3 & & 4 & & 7 & & 0 & & 11 & & 8 \\ & e & & f & & e & & f & & e & & f & \\ 1 & & 6 & & 5 & & 2 & & 9 & & 10 & & 1 \end{array}$$

In general consider the following double rows, and take every other square in each double row.

$$\begin{array}{ccccccc}
0 & 3 & 4n-4 & 7 & \dots & 4n-1 & 0 \\
1 & 4n-2 & 5 & 4n-6 & \dots & 2 & 1 \\
4 & 3 & 0 & 7 & \dots & 4n-1 & 4 \\
1 & 2 & 5 & 4n-2 & \dots & 6 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
4n-4 & 3 & 4n-8 & 7 & \dots & 4n-1 & 4n-4 \\
1 & 4n-6 & 5 & 4n-10 & \dots & 4n-2 & 1
\end{array}$$

4. Construction of a Class of Clean Triangulations

We start by taking $4n$ copies of the surface S_{n^2-2n+1} with $K_{2n,2n}$ embedded on it. Call these copies C_0, C_1, C_2, ..., C_{4n-1}.

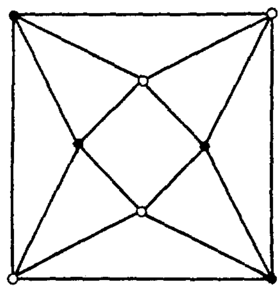

Fig. 7

If s and t, for $0 \le s, t \le 4n-1$, have different parity, then we do the following operation. We choose a representative class of squares in C_s and the corresponding representative class of squares in C_t. Now we take the first square in the class on C_s and excise it from the surface, leaving a hole. Do the same with the first square in the class on C_t. Then we connect the two holes by a handle in the following way. Take the triangulated portion of *Fig. 7,* and identify the outer square with the boundary of the hole in C_s and the inner square with the boundary of the hole in C_t. We do this in such a way that we obtain an orientable surface. We will call this a *handle operation.* Notice that a handle operation replaces two squares by eight triangles. We repeat the handle operation until each of the n squares in the class in C_s has been connected to the corresponding square in the class in C_t.

We continue in this manner until every C_s with s even has a class of squares joined to a class of squares on every C_t with t odd. The reader should realize that we use the complete bipartite graph $K_{2n,2n}$ again; now the vertices are the copies of $K_{2n,2n}$ embedded in S_{n^2-2n+1}. See *Fig. 8* for $n = 2$. The four copies C_0, C_2, C_4, and C_6 are in the horizontal row, and the other four are in the vertical row.

The result of the above procedure is a triangulation of an orientable surface S. We can compute the genus of S as follows. We started with $4n$ copies of S_{n^2-2n+1}, then added $4n^3$ handles. The first $4n-1$ added handles connected the surfaces

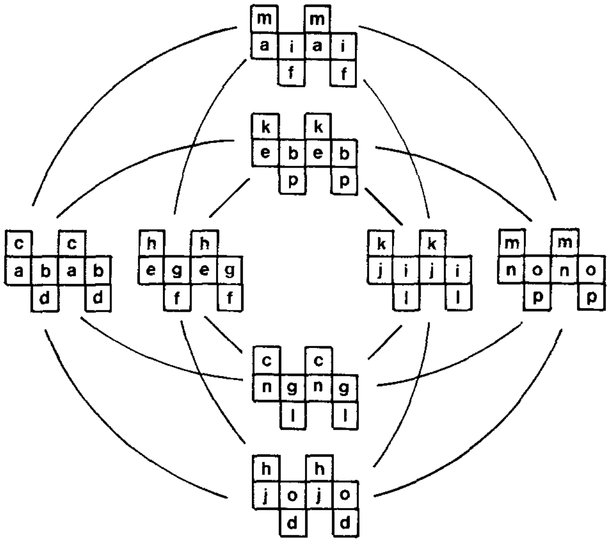

Fig. 8. Plan for a clean triangulation of S_{33}.

together, adding nothing to the total genus. So the net addition to the genus from the handle operations is $4n^3 - 4n + 1$. Thus the genus of S is

$$4n(n^2 - 2n + 1) + 4n^3 - 4n + 1 = 8n^3 - 8n^2 + 1.$$

Now we compute how many triangles are in the triangulation. Each copy of the embedding of $K_{2n,2n}$ has $2n^2$ squares. There are $4n$ copies, for a total of $8n^3$ squares. When we do a handle operation, we replace two squares by eight triangles. Thus there are $32n^3$ triangles in the triangulation.

In our construction we lost all the squares as faces, but their boundaries are still there, so we may still call these boundaries squares. Also in each copy C_s the 1-skeleton still exists and may still be called C_s.

Now we shall prove that the constructed triangulation is clean. Assume there is an extra triangle with vertices x_1, x_2, x_3. We consider three cases.

1) If all three vertices belong to one copy C_s then they cannot form a triangle because the 1-skeleton C_s is bipartite.

2) Let x_1 and x_2 be incident with C_s and x_3 with C_t and $s \neq t$. The edge (x_1, x_3) can only have been created by a handle H containing C_s and C_t. The edge (x_1, x_2) is also created by a handle H' connecting C_s and C_t. H must equal H', because any two different handles connecting C_s and C_t have no vertices in common, since the representative classes are disjoint. So all these vertices x_1, x_2, x_3 belong to one handle. Thus they are three of the eight vertices in *Fig. 7.* Obviously if they form a triangle it is not an extra one.

3) If the three vertices are incident with three different copies, say C_s, C_t, C_r, then two of the numbers s, t, r have the same parity, say t and r. So C_t and C_r are

not connected by a handle, and the corresponding two vertices are not connected by an edge.

Thus no extra triangles exist, and the construction produces a clean triangulation.

5. The Main Theorem

We are ready to prove the limit theorem which we mentioned in the introduction.

Proof. Let $p \geq 1$. We choose n to be an integer as small as possible such that

$$p \leq 8n^3 - 8n^2 + 1.$$

Then obviously we have

$$8n^3 - 32n^2 + 40n - 15 < p \leq 8n^3 - 8n^2 + 1.$$

If p is equal to $8n^3 - 8n^2 + 1$, then we have already exhibited a clean triangulation of S_p with $32n^3$ triangles, so

$$\tau(S_p) \leq 32n^3.$$

Now suppose that p is not equal to $8n^3 - 8n^2 + 1$. We then make the following modification to the construction of the clean triangulation above. We only perform the handle operation until we have a surface of genus p. Where a handle would have been, we triangulate the two squares by adding an extra vertex in the center of each square and connecting all four vertices of the square to it. This adds eight triangles for each pair of squares. Clearly no extra triangles are created, since each additional vertex is adjacent only to the four vertices of the square. There are still $32n^3$ triangles in the triangulation, no extra triangles appear, and the inequality

$$\delta(S_p) \leq \tau(S_p) \leq 32n^3$$

holds. Therefore we have

$$\frac{\delta(S_p)}{p} \leq \frac{\tau(S_p)}{p} \leq \frac{32n^3}{p} \leq \frac{32n^3}{8n^3 - 32n^2 + 40n - 15},$$

and hence

$$4 = \lim_{p\to\infty} \frac{\delta(S_p)}{p} \leq \lim_{p\to\infty} \frac{\tau(S_p)}{p} \leq 4.$$

To prove the theorem for nonorientable surfaces, we modify the construction slightly. In one handle operation, we make the identification of one square backwards, thus making the entire surface nonorientable. The rest of the argument is then the same. ∎

One can ask what the clean chromatic number of a surface would be; that is, how many colors are sufficient to color the vertices of every clean triangulation of a surface S. If we denote the clean chromatic number of S by $\lambda(S)$, then the inequality $\lambda(S) \leq \chi(S)$ holds where $\chi(S)$ is the chromatic number of S. If S is not the sphere, then $\lambda(S) \leq \chi(S) - 1$, because Dirac [1, 2] has proven that every graph on S with chromatic number $\chi(S)$ contains a clique with $\chi(S)$ vertices and therefore has extra triangles.

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Nora Hartsfield

University of California
Santa Cruz
CA 95 064, U.S.A.

and

Western Washington University
Bellingham
WA 98 225 U.S.A.
frog@nessie.wwu.edu

Gerhard Ringel

University of California
Santa Cruz
CA 95 064, U.S.A.